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The Complexity of Facets (and Some Facets of Complexity)

C. H. PAPADIMITRIOU

Stanford University, Stanford, California 94305 and National Technical University of Athens, Athens, Greece

AND

M. YANNAKAKIS

Bell Laboratories, Murray Hill, New Jersey 07974

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1. INTRODUCTION

Many important combinatorial optimization problems, including the traveling salesman problem (TSP), the clique problem, and many others, call for the optimization of a linear functional over some discrete set of vectors. In the case of the TSP, for example, we wish to minimize $c'x$ subject to $x \in F$. Here $x = (x_{ij})$ is an $\binom{n}{2}$-dimensional variable vector, indexed by the edges of K_n, the complete graph with n nodes; c is the $\binom{n}{2}$-dimensional vector of the inter-city distances, and $F \subseteq Z^{\binom{n}{2}}$ is the set of characteristic vectors of the tours of n cities (considered as subsets of the edges of K_n, the complete graph with n nodes). In the case of the clique problem, we wish to maximize $c'x$ subject to $x \in F$, where x is an n-dimensional variable vector, c is the vector of n ones, and F is the set of characteristic vectors of the cliques of the given graph.

Now, these optimization problems are equivalent to

$$\begin{aligned} &\text{optimize} \quad c'x \\ &\text{subject to } x \in CH(F), \end{aligned} \tag{1}$$

where $CH(F)$ is the *convex hull* of F. $CH(F)$ is called the *TSP* (resp. *clique*) *polytope*. Since optimizing a linear functional over a convex polytope is what linear programming is all about [D, PS], it would appear that formulation (1) brings the TSP (and a host of other combinatorial optimization problems) within the realm of realistic solution. All that is needed to this end is a characterization of the *facets* of $CH(F)$, i.e., the nonredundant system of linear inequalities that define this polytope.

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Characterizing the facets of the TSP polytope is one of the most well-studied problems in combinatorics, being under constant attack over 30 years. Intuitively, a facet describes a constraint that all tours must satisfy. The constraint must be in a sense severe, that is, there must exist an affinely independent set of d tours (where d is the dimension of the polytope) that satisfy it with equality. For example, one of the easiest class of facets of the TSP is the one that states that each city is entered and departed from exactly once:

$$\sum_{j=1}^{n} x_{ij} \leqslant 2,$$
$$\sum_{j=1}^{n} x_{ij} \geqslant 2, \qquad i = 1,..., n.$$

Another easy (though not obvious) class are the so-called *subtour elimination constraints*

$$\sum_{i \in S\, j \notin S} x_{ij} \geqslant 2, \qquad \varnothing \neq S \subsetneq \{1, 2,..., n\}.$$

Since the discovery of these two classes of constraints [DFJ] several new classes of facets have been discovered, in a futile attempt at obtaining a complete characterization. Chvàtal [C1] described an important class called *comb constraints* subsequently generalized by Grötschel and Padberg [GP]. More recently, Grötschel [G] showed how to derive a new TSP facet for each *maximal Δ-hypohamiltonian* graph. A graph is hypohamiltonian if it has no Hamilton circuit, but all subgraphs induced by deleting any node and its incident edges do. It is called *Δ-hypohamiltonian*, if, furthermore, it has property Δ, a complex technical property which states roughly that there are enough Hamilton circuits in the induced subgraphs that cover all the edges of the graph. Starting from a such a graph now, add all edges you can without creating a Hamilton circuit; the resulting graph G is a maximal Δ-hypohamiltonian graph. Grötschel showed that the inequality stating that each tour cannot contain more than $n-1$ edges of G is a facet (it is obviously a valid inequality). Δ-hypohamiltonian graphs are such a delicate concept that it takes arguments even to establish that there are infinitely many of them [DV, L, Th]. It is not known whether they form a dense set. As a byproduct of the results in this paper, we obtain a *new, much more natural* class of TSP facets.

Analogous work has been done for the clique and several other polytopes (see [Ba, BZ, C2, NT, P1, P2, W] among others). In view of the computational complexity of the TSP, clique, and the other problems [K], it is natural to ask whether there are negative complexity results concerning the facets of the associated polytopes. In other words, we wish to study, for example, the complexity of TSP FACETS, the problem of recognizing whether a given inequality is a facet of the TSP polytope. Indirect information concerning the complexity of this problem was obtained in [KP]. It was shown that if TSP FACETS $\in NP$, then $NP = \text{co}\,NP$. Is there, however, a complexity class, probably above NP, which is the natural niche of this language?

In this paper we introduce such a class. It is the class of all languages that are the intersection of a language in NP and a language in $\text{co}\,NP$ (not the same as

$NP \cap \text{co}\, NP$!) This class, which we denote D^p, is the polynomial-time analog of the *difference* class of languages in recursion theory. Alternatively, D^p can be defined as the class of all predicates $R(x)$ that can be expressed as $R(x) = [\exists y P(x, y)] \wedge [\forall z Q(x, z)]$, where P and Q are predicates which are polynomially balanced (in the sense that they are true only if the size of y and z is bounded by a polynomial in the size of x) and can be checked in polynomial time. D^p contains both NP and co NP and is contained in Δ_2^p (polynomial time with a SATISFIABILITY oracle). Several interesting classes of problems are in D^p:

(1) *Facets.* It is not hard to see that the facets of the polytopes that correspond to the common combinatorial optimization problems (such as TSP and clique) are in D^p.

(2) *Critical problems.* Some of the hardest problems in graph theory concern *critical graphs*, that is, graphs that do not have a property but deletion (sometimes addition) of an edge (or node) creates a graph with the property. For example, graphs that are hypohamiltonian, maximal nonHamiltonian, minimal non-3-colorable, etc. In fact, traditionally in graph theory, characterization of a property means finding the critical graphs for this property (cf. Kuratowski's characterization of planar graphs. Beineke's characterization of line-graphs, the Strong perfect graph conjecture, and a large number of other results and conjectures—see, e.g., [Be, H]). There are also critical problems that are not graph-theoretic; for example, unsatisfiable Boolean formulae for which deletion of any clause creates a satisfiable formula. It is easy to see that if the property is NP-complete (e.g., Hamiltonian, 3-colorable graph, satisfiable formula), then the corresponding critical problem is not in NP unless $NP = \text{co}\, NP$.

(3) *Exact problems.* For example, the set of all pairs (G, k) such that the maximum clique of G has size *exactly* k. Such problems also belong to D^p. It was shown in [LM] that if this problem is in $NP \cup \text{co}\, NP$, then $NP = \text{co}\, NP$.

(4) *Unique solution problems.* For example, the set of all Boolean formulae that have exactly one satisfying truth assignment. Such problems have been studied in the past; they are easily seen to be co NP-hard.

Problems of all these kinds are in D^p, and are not known (or believed) to be in a class below D^p. Whether problems in these classes are complete for D^p, is therefore an interesting question. Notice that natural problems complete for classes "between" NP and PSPACE are extremely rare.

We show that the exact clique problem referred to in class (3) above is complete for D^p. As a consequence, so is the exact-cost version of any combinatorial optimization problem, such as the TSP, to which CLIQUE is polynomial-time transformable, as long as the transformation also transforms costs in a strictly monotone way (as is the case with most known transformations). This provides a natural class of problems complete for D^p. Furthermore, we show that recognition of the facets of the clique polytope is also complete for D^p, thereby establishing that D^p is indeed the right complexity class for facets.

From the family of critical problems, we show that irreducible posets (that is, partial orders that are critical with respect to dimension) are NP-hard, thereby proving that they must be above $NP \cup \text{co}\, NP$ if $NP \neq \text{co}\, NP$. We have not been able to show that TSP FACETS or any of the critical problems is D^p-complete. This difficulty seems to reflect the extremely delicate and deep structure of critical problems—too delicate to sustain any of the known reduction methods. One way to understand this is that exhibiting critical graphs is usually the object of hard theorems. In contrast, a proof of completeness (e.g., reduction from a satisfiability-like problem) would create a virtual "factory"of critical graphs, something which is intuitively quite difficult. We can show, however, certain polynomial-time transformations *between* critical problems. For example, the undirected version of the maximal (wrt edges) non-Hamiltonian graph problem is polynomially transformable to the directed one; both problems transform to *critical integer programming* (unsatisfiable linear systems of inequalities in integer variables that become satisfiable if we omit any inequality). More importantly, the maximal non-Hamiltonian graph problem transforms to TSP FACETS. As a consequence of our proof, we can construct TSP facets for $3n$ cities starting from maximal non-Hamiltonian graphs on n nodes. This is a far more general and natural class of facets than those of the previous champion [G] which were based on maximal Δ-hypohamiltonian graphs.

We finally show a related negative result concerning the TSP polytope. Given a rational point p in $\binom{n}{2}$-space, one may wish to *generate a separating hyperplane* between p and the polytope, if this is possible, or otherwise to report that p is in the polytope—and hence no such hyperplane exists. It was shown in [KP] (and independently in [GLS]) that this cannot be done in polynomial time unless $P = NP$. Both proofs used the ellipsoid algorithm for linear programming [Kh]. We strengthen this result (for the TSP) in two ways: we show that just *testing whether p is in the polytope* is NP-complete, and prove it *without making use* of the ellipsoid algorithm.

The rest of the paper is organized as follows. In Section 2 we define D^p, list some representative problems, and show the completeness of the exact clique problem. In Section 3 we prove the completeness of the clique facets. In Section 4 we examine critical problems and the relationship to the TSP facets. Finally, in Section 5 we prove the NP-completeness of the interior point problem.

2. The Class D^p

Define $D^p = \{L_1 \cap L_2 : L_1 \in NP, L_2 \in \text{co}\, NP\}$. Since Σ^* is in both NP and $\text{co}\, NP$, we have NP, $\text{co}\, NP \subseteq D^p$. Also trivially, $D^p \subseteq \Delta_2^p$ (where Δ_2^p is defined as P^{NP}). D^p contains a host of natural problems, whose membership in $NP \cup \text{co}\, NP$ is in serious doubt.

(1) TSP FACETS. "Given an inequality $a'x \leqslant b$ with integer coefficients and $\binom{n}{2}$ variables, is it a facet of $CH(\bar{F})$?" As is customary in this area, we include in $\bar{F}$ all subsets of tours, so that the TSP polytope will be full-dimensional.

(2) *Clique facets.* "Given a graph G and an inequality, is the inequality a facet of the clique polytope of G?"

(3) *Exact* TSP. "Given an $n \times n$ distance matrix c_{ij} and an integer L, is it true that the optimum TSP tour has cost L?"

(4) *Exact clique.* "Given a graph G and a number k, is it true that the largest clique of G has size exactly k?"

(5) *SAT-UNSAT.* "Given two Boolean Formulae F, F', is it true that F is satisfiable whereas F' is not?"

(6) TSP *supporting hyperplane.* "Given an inequality $a'x \leqslant b$ with integer coefficients and $\binom{n}{2}$ variables is it a valid inequality which has a nonempty intersection with the TSP polytope?"

(7) *t-irreducible poset.* "Given a poset P, is it true that it has dimension t, but every subposet of it has dimension at most $t-1$?"

(8) *Maximal non-Hamiltonian graph* (MNHG). "Given graph G, is it true that G has no Hamilton circuit, but adding any edge to it creates one?"

(9) *Maximal non-Hamiltonian digraph* (MNHD).

(10) *Critical integer programming.* "Given a system $Ax \leqslant b$ is it true that it has no integer solution, but omitting any single inequality permits a solution?"

(11) *Unique SAT.* "Given a Boolean formula F, is it true that it has one and only one satisfying truth assignment?"

PROPOSITION 1. *Problems* (1)–(11) *are in* D^p.

Proof. All of these problems can be rephrased as the conjunction of a predicate in NP and a predicate in co NP. We exemplify this with problems (1) and (11), the only slightly nontrivial cases. For (1) the co NP predicate states that all tours satisfy the given inequality. The NP predicate states that there exists a set of $\binom{n}{2}$ tours that are affinely independent (as characteristic vectors) and satisfy $a'x = b$. For (11) the NP part is satisfiability, and the co NP predicate states that, for all pairs of distinct truth assignments, not both satisfy F. ∎

It is straightforward that at least one of these languages is complete for D^p.

LEMMA 1. *SAT-UNSAT is complete for* D^p.

Proof. Starting from an instance x of any problem A in D^p, we construct two Boolean formulae $F(x), F'(x)$, one for the NP predicate of A and one for the co NP predicate. ∎

We can now show that D^p has a "natural" complete language.

THEOREM 1. *Exact clique is* D^p*-complete.*

Proof. Reduce *SAT-UNSAT* to it. Starting from (F, F'), we can now construct two graphs G and G' and two integers k, k' such that the maximum clique of G is of size k if F is satisfiable, and of size $k-1$ otherwise, and similarly for G'. It is easy to see that this holds for standard transformations from *SAT* to *clique* [PS]. By standard padding constructions we can make sure that $k \neq k'$. Construct now the product $G \times G'$ of G and G' (a copy of G for each node of G', a complete bipartite graph for each edge of G' [GJ]). It is not hard to see that $G \times G'$ has a maximum clique of size exactly $k(k'-1)$ iff F is satisfiable and F' is not. ∎

THEOREM 2. TSP *supporting hyperplane is D^p-complete.*

Proof. We first show that *exact* TSP is D^p-complete. To see this, we reduce *SAT-UNSAT* to *exact* TSP. Starting from a pair (F, F') of formulae we create G, G' and two pairs of nodes (s, t) of G and (s', t') of G' such that there is a Hamilton path from s to t (resp. s' to t') iff F (resp. F') is satisfiable. Furthermore, by standard techniques we can carry out this construction so that in G' there are always *two* paths, one starting at s' the other ending at t', covering all nodes. Given these two graphs, we first identify s' with t, add a node x and the edges $[s, x]$, $[x, t']$ and call the resulting graph H. From H we create an instance of the TSP by assigning cost 0 to all edges of H, cost 1 to the nonedges of G', and cost 2 to all other pairs. It is easy to see that the optimum tour has cost 1 iff F is satisfiable and F' is not.

Once *exact* TSP is D^p-complete, the completeness of TSP *supporting hyperplane* follows: the optimum tour has cost exactly L iff $-c'x \leqslant -L$ is a supporting hyperplane of the TSP polytope. ∎

The rest of the problems (1)–(11) will be discussed further in the next sections. We note here that the precise complexity of *unique SAT* is a persistent open question.

3. THE CLIQUE FACETS

At first let us note that there is some kind of duality between the TSP and the *clique* polytopes: in the TSP we have only one polytope for every n (number of cities); the input (the distances) is reflected in the function that has to be optimized. In the case of the clique problem the input is reflected in the polytope—there is one polytope for each graph; on the other hand, the functional that has to be optimized is fixed (all coefficients are 1). We are going to show the D^p-completeness of the clique facet problem for a facet that is parallel to this (fixed) optimization functional.

THEOREM 3. *Clique facet is D^p-complete.*

Proof. We reduce *exact clique* to it. Given a graph G and a number k, we shall construct another graph H such that the inequality $\sum_{i=1}^{n} x_i \leqslant k$ is a facet of the polytope $P(H)$ that corresponds to H if and only if the maximum clique of G has size exactly k. The graph H is constructed from G as follows. For every pair u, v of nonadjacent nodes of G we add a set of $2k-1$ new nodes $S(u, v) = \{a_{uv1}, \ldots, a_{uv,2k-1}\}$. We also add the edges $\{[a_{uvi}, u], [a_{uvi}, v]$: all $i\}$ and $\{[a_{uvi}, a_{uvj}]$:

$i \neq j \pm 1 \bmod 2k-1)\}$. Thus each such set of new nodes induces in H the complement of a cycle of length $2k-1$. This completes the construction of H.

We claim that the inequality $\sum x_j \leqslant k$ is a facet of $P(H)$ iff the largest clique of G has size k. First, if G has a clique of size $k+1$, then the inequality is not valid. Suppose now G has no clique of size k. Then the only cliques of size k of H (equivalently, vertices of $P(H)$ on the hyperplane $h = \{x: \sum x_j = k\}$) consist of $k-1$ new nodes and one node of G. Thus they lie on the hyperplane $\sum_{j \in G} x_j = 1$. Since the vertices that lie on the hyperplane $\sum x_j = k$ lie also on another hyperplane, we conclude that either they are not affinely independent or there are not sufficiently many of them. It follows that $\sum x_j = k$ is not a facet.

Conversely, if G has a clique of size k, then this is also a clique of H, and thus a vertex on the hyperplane h. We claim that this set of vertices is now affinely independent. Suppose it is not; suppose that it satisfies another equation, say

$$\sum \alpha_j x_j = k \tag{$*$}$$

We shall prove that all α_j's are equal to 1, and thus the equation is not new. First, all the α_j's corresponding to the nodes of $S(u, v)$ must be equal. For example, let us show that $\alpha_{a_{uv1}} = \alpha_{a_{uv2}}$. It suffices to exhibit two cliques of size k having $\{a_{uv1}, a_{uv2}\}$ as their symmetric difference (just substitute the two vertices corresponding to the two cliques into equation $(*)$). These cliques are: $\{u, a_{uv1}, a_{uv4}, a_{uv6},..., a_{uv,2k-2}\}$ and $\{u, a_{uv2}, a_{uv4},..., a_{uv,2k-2}\}$. Similarly, $\alpha_u = \alpha_v$ for all nonadjacent vertices u, v of G—and thus for *all vertices of* G (without loss of generality, the complement of G is connected). Now the clique of G establishes that all α_u's are equal to 1. Substituting into $(*)$ a clique of H that consists of u and $k-1$ nodes of $S(u, v)$ we deduce that all other α's are also equal to 1. The proof is complete. ∎

COROLLARY. *Recognizing the facets of node packing, set packing, node covering, set covering polytopes is D^P-complete.*

Proof. The standard reductions preserve the facial structure of the polytopes. ∎

4. CRITICAL PROBLEMS AND TSP FACETS

The *dimension* of a poset P is the minimum number d of linear orders $L_1,..., L_d$ whose intersection is P; i.e., an element x precedes another element y in P iff x precedes y in all the linear orders $L_1,..., L_d$. A poset is *d-irreducible* if it has dimension d but deletion of any element results in a poset of dimension $d-1$ (for more information see [Go]).

THEOREM 4.5. *5-irreducible poset is NP-hard.*

Proof. The reduction relies on a recent strong result of Trotter and Ross [TR], which states that every poset of dimension t is contained in some $(t+1)$-irreducible poset, for every $t \geqslant 3$. More specifically, when given a poset P of dimension t and a linear extension L_1 of P which belongs to a minimum realizer of P (i.e., such that there are linear extensions $L_2, \ldots, L_t$ with $\bigcap_{i=1}^{t} L_i = P$), they can construct (in polynomial time) another poset P' which contains P and is $(t+1)$-irreducible.

In [Y] the NP-completeness of the dimension t problem was shown for all $t \geqslant 3$. Furthermore, a reduction was given there to the dimension 4 problem for partial orders of height 1, where the constructed poset P had the property that if its dimension was 4, then a minimum realizer contained a given linear extension L_1. The result now follows by applying the construction of [TR] on the posets produced by the reduction of [Y]. If P has dimension 4 then the resulting poset P' is 5-irreducible. On the other hand, if P' is 5-irreducible, deletion of any element leaves a 4 dimensional poset. Since P' contains P, this means that P has dimension 4. For the (complicated) details of the construction, the reader is referred to [TR] and [Y]. ∎

A complete list of 3-irreducible posets (9 infinite families and 18 odd examples!) is given in [Ke, TM].

COROLLARY. *5-irreducible poset is not in* $NP \cup \text{co}\, NP$ *unless* $NP = \text{co}\, NP$.

We do not know if any of the critical problems is D^P-complete. The difficulty arises from the fact that the constructions will have to be extremely delicate, and require a better understanding of critical graph theory. Also, most of the reduction methods (e.g., the ones employing "gadgets") do not even preserve criticality, let alone create a critical graph from an arbitrary (noncritical) one. We mention now some cases where criticality can be preserved.

PROPOSITION 2. *MNHG is transformable to MNHD.*

Proof. The standard reduction from the undirected to the directed Hamilton problem (replace every edge by two opposite arcs) happens to work. ∎

PROPOSITION 3. *MNHD is transformable to critical integer programming.*

Proof. Given a graph G with n nodes we construct a set of inequalities in nonnegative integer variables. The variables are of the form x_{ij} with intended meaning "$x_{ij} = 1$ iff the jth node in the Hamilton circuit is node i." The inequalities are

$$\sum_{j=1}^{n} x_{ij} \leqslant 1, \quad i = 1, \ldots, n \text{ (every node appears at most once in the tour)}, \tag{2}$$

$$\sum_{i=1}^{n} x_{ij} \geqslant 1, \quad j = 1, \ldots, n \text{ (some node is the } j\text{th node of the tour)}, \tag{3}$$

$$x_{ij} + x_{k,j+1} \leqslant 1, \quad j = 1, \ldots, n \text{ and } [i, k] \text{ is not an edge of } G \text{ (here } n+1 \text{ means } 1) \tag{4}$$

(the tour uses only edges of G).

We claim that system (2)–(4) is critical iff G is maximal non-Hamiltonian. If G has no Hamilton circuit, then obviously (2)–(4) has no solution. If furthermore G is maximal, if we omit any of the inequalities in (4) then (2)–(4) does have a solution. Also, if we omit an inequality from (3), then there does not have to be a jth node in the circuit, and thus a solution can be constructed from any Hamilton path of G, by placing one of its end points jth. Finally, if we omit an inequality in (2) then i can appear many times on the "circuit," in particular, we can set $x_{ij} = 1$ for all j, and all other variables 0. Hence (2)–(4) is critical.

Conversely, if (2)–(4) is critical, it follows that by omitting any constraint among those in (4), i.e., by adding any edge, we have a Hamilton circuit. Hence, G is maximal nonHamiltonian. ∎

In the rest of this section we will relate MNHG to TSP FACETS. Consider an inequality $a'x \leqslant n-1$, where a is an $\binom{n}{2}$-dimensional vector of 0's and 1's. With a we can associate the graph G whose adjacency matrix is listed in the vector a. Let $x(G)$ denote the sum of those x_{ij} for which G contains the arc(i, j). Then we can write the inequality as $x(G) \leqslant n-1$. Note that the facets described in [G] have this form. A node u of G is called a *supernode* if it is adjacent to every other node.

PROPOSITION 4. *If* $x(G) \leqslant n-1$ *is a facet of the* TSP *polytope, then* G *is a maximal nonHamiltonian graph with no supernodes.*

Proof. If G is Hamiltonian, then $x(G) \leqslant (n-1)$ is not a valid inequality, and thus certainly not a facet. So, suppose G is not Hamiltonian, but neither is G with edge e added to it. Then it is easy to see that all vertices of the TSP polytope on the hyperplane $x(G) = n-1$ (i.e., all tours which correspond to a Hamilton path of G) *cannot* contain edge e. So, they all lie on the hyperplane $x_e = 0$. Therefore, either they are not affinely independent or there are not sufficiently many of them; thus, $x(G) \leqslant n-1$ is not a facet. Similarly, if G has a supernode, then two edges incident upon the supernode participate in any vertex on the hyperplane, and so these vertices are not affinely independent. ∎

The converse to the proposition does not hold; i.e., for a graph G to define a facet much more is required than just satisfying the condition of the proposition. We are going to show, however, that from any such graph H we can construct a graph G with $3n$ nodes that defines a facet.

Let $H = (V, A)$ be a graph with no supernodes. We shall construct a graph $G = (N, E)$ with $3n$ nodes such that the inequality $x(G) \leqslant 3n-1$ is a facet of the TSP polytope iff H is maximal non-Hamiltonian. For each node u of H, G has 3 nodes u_1, u_2, u_3 connected in a cycle u_1–u_2–u_3–u_1. In addition G has the following edges:

(1) All edges $[u_3, v_3]$,

(2) For each edge $[u, v]$ of H, the edges $[u_3, v_1]$, $[v_3, u_1]$.

(3) The edge $[u_3, v_2]$ if u dominates v in H, where we say that u dominates v if u is adjacent to v and to every node adjacent to v.

LEMMA 1. *G has a Hamilton circuit iff H has one.*

Proof. If H has a Hamilton circuit $(v^1, v^2,..., v^n, v^1)$ then so does G: $[v_1^1, v_2^1, v_3^1, v_1^2,..., v_3^n, v_1^1]$. Suppose now that G has a Hamilton circuit C. We shall show that H has one also. The proof is by induction on the number of edges of the form $[u_3, v_2]$ in C. First, suppose that there are no such edges. Then we claim that there are no $[u_3, v_3]$ edges either. Because, suppose C contains $[u_3, v_3]$. Then it must contain the whole segment $[u_1, u_2, u_3, v_3, v_2, v_1]$. Now from the endpoints of this segment we must go to a w_3 node, then to w_2, w_1 and so on. There is no way to close the cycle. Thus, there are only edges of the form $[u_3, v_1]$; this excludes the $[u_3, u_1]$ edges also. But this leaves us with a circuit of the form $[v_1^1, v_2^1, v_3^1, v_1^2,..., v_3^n, v_1^1]$ which corresponds to a Hamilton circuit of H.

Next, for the induction step, suppose that G has a Hamilton circuit C with some edge of the form $[u_3, v_2]$. We shall show how to construct from C another Hamilton circuit, with (at least) one fewer such edge, and so the proof will be completed.

The proof is based on a case analysis, depending on the next node w in C, after v_2, and whether $[v_3, v_1] \in C$.

Case 1. $w = v_3$, $[v_3, v_1] \notin C$. Then v_1 must be picked up by C somehow, thus $C = [...u_3, v_2, v_3,..., x_3, v_1, y_3,...]$. The new Hamilton circuit is $C' = [...u_3, v_1, v_2, v_3,..., x_3, y_3,...]$. (Since u dominates v, $[u, v] \in A$ and, therefore, $[u_3, v_1] \in E$.)

Case 2. $w = v_3$, $[v_3, v_1] \in C$. Then $C = [...u_3, v_2, v_3, v_1, x_3,...]$. $C' = [...u_3, v_1, v_2, v_3, x_3,...]$.

Case 3. $w = v_1$, $[v_3, v_1] \notin C$. Then C has the form $[..., u_3, v_2, v_1, x_3,..., y_j, v_3, z_i,...]$. Then $C' = [..., u_3, y_j,..., x_3, v_1, v_2, v_3, z_i,...]$. Here we used the fact that, if $[u_3, v_2] \in E$ (and thus u dominates v) and $[v_3, y_j] \in E$, then, certainly, $[u_3, y_j] \in E$. If $j = 1$ this follows from the definition of dominance; if $j = 2$ it follows from the transitivity of dominance; if $j = 3$ it always holds.

Case 4. $w = v_1$, $[v_3, v_1] \in C$. Then $C = [...u_3, v_2, v_1, v_3,...]$ and $C' = [...u_3, v_1, v_2, v_3,...]$.

Case 5. $w = x_3$, $[v_3, v_1] \notin C$. Then $C = [...u_3, v_2, x_3,..., y_j, v_3, z_i,..., s_3, v_1, t_3,...]$, since the nodes v_1 and v_3 must be picked up somehow (the order is not a loss of generality). $C' = [...u_3, v_1, v_2, v_3, y_j,..., x_3, z_i,..., s_3, t_3,...]$. Again, $[x_3, z_i] \in E$ because x dominates v and $[v_3, z_i] \in E$.

Case 6. $w = x_3$, $[v_3, v_1] \in C$. $C = [...u_3, v_2, x_3,..., y_3, v_1, v_3, z_i,...]$. (Again, since the argument is symmetric in $u_3 - x_3$, it does not matter if we reverse the last part of C). Then $C' = [..., u_3, x_3,..., y_3, v_1, v_2, v_3, z_i,...]$.

The proof is complete. ∎

LEMMA 2. *If H is not maximal non-Hamiltonian then neither is G.*

Proof. If H has a Hamilton circuit, then so does G by Lemma 1. Suppose that H has no Hamilton circuit, but neither does H with some edge $e = [a, b]$ added. Add the edge $[a_3, b_1]$ to G; we show that if a Hamilton circuit C is created then H must be Hamiltonian, a contradiction. Eliminate from C as many edges of the form $[u_3, v_2]$ (with $u \neq v$) as possible, applying the operations in the proof of Lemma 1. If no edge of the form $[u_3, v_2]$ is left in the resulting circuit C', then C' corresponds to a Hamilton circuit of the graph obtained by adding $[a, b]$ to H. So assume that some edge $[u_3, v_2]$ remains. Going again through the different cases in the proof of Lemma 1, we observe that the following properties of the dominance relation are used in carrying out the operations: in Cases 1, 2, and 4 we use the fact that if $[u_3, v_2] \in E$ then $[u, v] \in A$ and, therefore, $[u_3, v_1] \in A$. This property remains valid. In Case 6 no property is used. In Case 5 we replaced a circuit of the form $[\alpha, u_3, v_2, x_3, \beta, y_j, v_3, z_i, \gamma, s_3, v_1, t_3, \delta]$ by $[\alpha, u_3, v_1, v_2, v_3, y_j, \beta', x_3, z_i, \gamma, s_3, t_3, \delta]$, where α, β, γ, δ are sequences of nodes and β' is the reverse sequence of β. Here we use the facts that $[u_3, v_2] \in E \Rightarrow [u_3, v_1] \in E$ and $[x_3, v_2] \in E$, $[v_3, z_i] \in E \Rightarrow [x_3, z_i] \in E$. The first fact remains true. The second one holds also unless $v = a$, $z_i = b_1$, and x is a node which dominates a in H but $[x, b] \notin A$. In this case $y_j \neq b_1$. Since u dominates v and $[v_3, y_i] \in E$ (because this is not the added edge) we have $[u_3, y_j] \in E$. Therefore, we can transform the circuit into $[\alpha, u_3, y_j, \beta', x_3, v_1, v_2, v_3, z_i, \gamma, s_3, t_3, \delta]$.

The only remaining case is Case 3. Here a circuit of the form $C = [\alpha, u_3, v_2, v_1, x_3, \beta, y_j, v_3, z_i, \gamma]$ is transformed into $[\alpha, u_3, y_j, \beta', x_3, v_1, v_2, v_3, z_i, \gamma]$. This transformation is still possible unless $v = a$, $y_j = b_1$, and u is a node which dominates a $(=v)$ in H but $[u, b] \notin A$. Therefore, after we eliminate as many $[u_3, v_2]$ edges as possible, we are left with a circuit $C = [\alpha, u_3, a_2, a_1, x_3, \beta, b_1, a_3, z_i, \gamma]$ with exactly one offending edge: $[u_3, a_2]$. Since for every $w \neq a$, w_2 is adjacent in C to w_1 and w_3, $\gamma\alpha$ corresponds to a path in H from a to u, and β corresponds to a path from a to b. Therefore, H has a Hamilton path from u to b of the form $(u, \alpha', \gamma', z, a, x, \beta, b)$. Since u dominates a, H has also a Hamilton path from a to b: $(a, z, \gamma, \alpha, u, x, \beta, b)$. Thus, adding the edge $[a, b]$ to H will create a Hamilton circuit. ∎

THEOREM 5. *$x(G) \leqslant 3n - 1$ is a facet of the* TSP *polytope iff H is maximal non-Hamiltonian.*

Proof. If H is not maximal non-Hamiltonian then neither is G (Lemma 2), and thus the inequality is not a facet by Proposition 4. Suppose now that H is maximal non-Hamiltonian and consider all vertices of the TSP polytope that lie on the hyperplane $h = \{x : x(G) = 3n - 1\}$. Every such vertex corresponds to a Hamilton path of G. Conversely, for every Hamilton path p of G there are two vertices that lie on the hyperplane h; the one vertex has 1's exactly in the coordinates that correspond to the edges of p, and the other one has, in addition, a 1 in the (missing) edge that connects the endpoints of p. Suppose that all these vertices satisfy another equation

$$\sum a_e x_e = b \tag{5}$$

We shall show that $a_e = 0$ if e is not an edge of G, and that all a_e for the edges e of G are equal. It will follow then that $x(G) = 3n - 1$ is the only equation satisfied by the vertices that lie on h.

To show that $a_e = 0$ for an edge e missing from G it suffices to show that the addition of e to G will create a Hamilton circuit. To see this substitute in (5) the two vertices of the polytope that correspond to a Hamilton path of G between the endpoints of e. Let us consider the different types of edges e that are not in G.

Case 1. $e = [u_3, v_1]$. Then $[u, v] \notin A$. Since H is maximal non-Hamiltonian there is a Hamilton path in H from v to u. This path corresponds in G to a Hamilton path from v_1 to u_3.

Case 2. $e = [u_1, v_1]$ or $[u_1, v_2]$ or $[u_2, v_2]$. Since H has no supernodes, there is a Hamilton path in H starting from u. Let $p = (u, \alpha, x, v, \beta, z)$ be such a path, where possibly $z = v$ or $u = x$. Then G has a Hamilton path that starts as u_1, u_2, u_3, covers the nodes in α, goes to x_3, then to z_3, then covers the nodes in β backwards, and finishes as v_3, v_2, v_1. To get a Hamilton path from u_1 to v_2, finish the previous path as v_3, v_1, v_2. For a path from u_2 to v_2 modify the beginning to u_2, u_1, u_3.

Case 3. $e = [u_3, u_2]$. Then u does not dominate v in H. That is, either $[u, v] \notin A$ or there is a node w such that $[v, w] \in A$ but $[u, w] \notin A$. In the first case consider a Hamilton path v–β–u of H from v to u. Then G has a Hamilton path v_2–v_1–v_3–$\bar{\beta}$–u_1–u_2–u_3, where $\bar{\beta}$ traverses the nodes corresponding to β in the same order. In the second case consider a Hamilton path of H from w to u. Let w–α–x–v–β–u be this path. Then G has the Hamilton path u_3–u_2–u_1–$\bar{\beta}'$–v_3–w_1–w_2–w_3–$\bar{\alpha}$–x_1–x_2–x_3–v_1–v_2, where $\bar{\beta}'$ traverses the nodes corresponding to β backwards, and $\bar{\alpha}$ traverses the nodes corresponding to α forwards.

We have shown thus far that $a_e = 0$ for all $e \notin E$. It remains to show that any two edges of G have the same coefficient in the equation.

We will do this as follows. We will exhibit a set of pairs (p_1, p_2) of Hamilton paths of G. The symmetric difference of the two paths p_1, p_2 of a pair will consist of two edges e_1, e_2. This implies that the two edges e_1 and e_2 have equal coefficients (just substitute the vertices corresponding to p_1 and p_2 into (5)). Consider a graph with the edges of G as nodes and draw an edge connecting e_1 and e_2. We will give enough pairs of Hamilton paths so that the graph constructed in this way is connected and spans all the edges of G.

(1) $[u_1, u_2]$, $[u_1, u_3]$, $[u_2, u_3]$ for all u. Let u be any node of H. Since H has no supernodes, there is a Hamilton path $[u, x, \ldots]$ that starts from u. This path corresponds to a Hamilton path $[u_1, u_2, u_3, \ldots]$ of G. If we delete $[u_2, u_3]$ and insert $[u_1, u_3]$, we get another Hamilton path of G. Associating with the Hamilton path of H starting from u the path of G $[u_3, u_2, u_1, x_3, x_2, x_1, \ldots]$, we can show similarly that the coefficients of $[u_1, u_2]$ and $[u_1, u_3]$ are equal.

(2) $[u_2, u_3]$, $[u_3, v_1]$ for all $[u, v] \in A$. Let u be any node of H and v any node adjacent to u. Take a Hamilton path of H starting from v. This corresponds to a

Hamilton path $p = [v_1, v_2, v_3, \ldots, u_1, u_2, u_3, \ldots]$ of G. If we replace $[u_2, u_3]$ by $[u_3, v_1]$ in p we get another Hamilton path of G.

(3) $[u_3, v_3]$, $[v_3, w_1]$ for all u, v, and some w. Let u, v be any two nodes of H. Take a Hamilton path of G ending in u_3: $p = [\ldots, v_1, v_2, v_3, w_1, \ldots, u_3]$. Replace $[v_3, w_1]$ by $[u_3, v_3]$.

(4) $[u_3, v_2]$, $[v_2, v_3]$ if u dominates v. As in 3, take a Hamilton path of G ending in u_3, and replace $[v_2, v_3]$ by $[u_3, v_2]$.

We have connected now enough pairs of edges to enable us to deduce that the coefficients of all the edges of G are equal. ∎

5. Interior Points

Consider, finally, the problem *interior points*. "Given a $\binom{n}{2}$-dimensional rational point p, is p in the TSP polytope?"

It is easy to see that *interior points* is in NP (if p is in the polytope, then by Caratheodory's theorem there exist at most $\binom{n}{2} + 1$ tours of which p is a convex combination). If TSP FACETS $\in NP$, then also *interior points* $\in$ co NP, as we could guess a separating facet for an interior point. The following result suggests that, unless $NP = \text{co } NP$, this is not the case.

THEOREM 6. *Interior points is NP-complete.*

Proof. Given a Boolean formula F with 3 literals per clause, we construct a graph $G = (V, E)$ which has a Hamilton circuit iff F is satisfiable. Constructions like this are well known [GJT, P1, PS]. However, our graph has a special property. Let us distinguish the subset $E_f \subseteq E$ of *forced* edges (one endpoint has degree 2). Then the number of edges *plus* the number of forced edges incident upon any node of G will be 4, and thus $|E| + |E_f| = 2\,|V|$. Furthermore, if $C \subseteq E$ is a Hamilton circuit of G, then so is $(E - C) \cup E_f$. Once we have such a graph G, the theorem follows, because we can construct a $\binom{|V|}{2}$-dimensional point p by letting

$$\begin{aligned} p_e &= 1 && \text{if } e \in E_f, \\ &= \tfrac{1}{2} && \text{if } e \in E - E_f, \\ &= 0 && \text{if } e \notin E. \end{aligned}$$

It is then easy to show that p is an interior point of the TSP polytope iff F is satisfiable. First, if p is an interior point then it must be the convex combination of *Hamilton circuits* of G because its coordinates add up to $|V|$. Conversely, if there is a Hamilton circuit C or G, then p is the midpoint of C and $(E - C) \cup E_f$, both considered as vertices of the TSP polytope. This would complete the proof.

We now proceed with our construction. It is another variant of that in [*PS*], showing that the Hamilton circuit problem is NP-complete. The construction is based

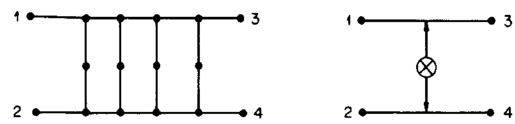

FIGURE 1

on certain graph-theoretic *gadgets*. For example, the graph shown in Fig. 1 has the property that, if it is the subgraph of the graph G, connected to the rest of G only via the nodes 1, 2, 3, and 4, then it behaves like two edges [1, 3] and [2, 4] connected by an *exclusive or*, so that any Hamilton circuit of G must contain exactly one of the two.

A variable x of F is simulated in G by a pair of parallel edges, as in Fig. 2. Finally a *clause* of F is simulated by the *three-input or* of Fig. 3a. If this structure is appended to G by its endpoints 1 and 2, then *at least one* of the three marked edges (corresponding to the three literals) must be traversed by any Hamilton circuit, and for every nonempty subset of these edges there is a Hamilton path from 1 to 2 traversing just these and no other marked edge. G contains a series of these clause graphs and a series of the variable graphs, with the marked edge of a clause graph corresponding to some literal connected via an exclusive or with the parallel edge of the variable subgraph which corresponds to the negation of this literal. Ordinarily, this would complete our construction, and it would not be hard to show that G has a Hamilton circuit iff F is satisfiable (see, e.g., [PS, P1] for this proof). Since we have special requirements for G, we must modify the clause graph, as shown in Fig. 3b. This version has two *sides*. There is an obvious vertical axis of symmetry between the nonforced edges in this graph; all pairs of images of this symmetry are connected by exclusive ors, not shown explicitly in the figure. Intuitively, the left side says "at least one literal is true," and the right side says "at least one is false." One of them is "turned off" by the Hamilton circuit, via the edges a, a', b, and b'; and the other one enforces its constraint (in the part of traversal shown in Fig. 3b, the left side is turned off). A side is turned off if the corresponding pair of edges among (a, a'), (b, b') is traversed (by the exclusive ors, either both a and a', or both b and b' are traversed), and the nodes of this side are "picked up" by the fragment of the Hamilton circuit between the two edges, whereas the remaining nodes are picked up as they would be in the one-sided version of Fig. 3a. It is not hard to see that, provided one of the two pairs is traversed, this is the only way for the two-sided three-input-or to be traversed by a Hamilton circuit.

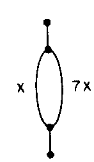

FIGURE 2

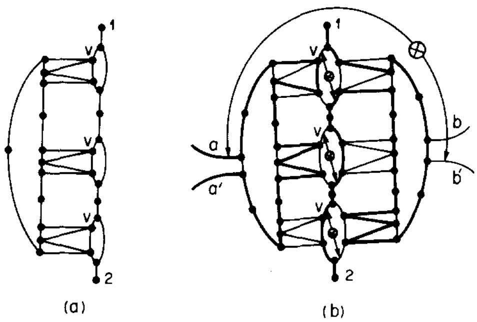

FIGURE 3

We connect the clause graphs in series, and we also identify the a' and b' edges of one with the a and b edges of the next. The a and b edges of the first clause have a common endpoint, and so do the a', b' edges of the last. Thus either all the right sides are turned off, or all the left sides are. Otherwise, the overall structure of G is the same as before; first come the two common endpoints of the a, b and a', b' edges, then a series of variable subgraphs, then the circle is closed with the series of clause subgraphs. It is not hard to see that all vertices of G have the desired property (the degree plus the number of incident forced edges equals 4).

Suppose G has a Hamilton cycle. It must first traverse either all a's or all b's, and it must also traverse the variable graphs, thus choosing a truth assignment. If the b's are traversed, then this truth assignment must satisfy at least one literal in every clause, and thus F is satisfiable. If the a's are traversed, then this truth assignment falsifies at least one literal in each clause, and thus the complementary truth assignment satisfies F. In any event, if C is the Hamilton circuit in question, then $(E - C) \cup E_f$ is the circuit corresponding to the complementary truth assignment.

Conversely, if F is satisfiable, then we can construct a pair of Hamilton circuits, corresponding to the satisfying truth assignment and its complement, so that their union is E and their intersection is E_f. ∎

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